

در آخذ

(عکس گرفتن از صفحه مجزوع)

وسط کتابش بچین نمید

ایمان میایتم 17 آبان ساعت 14:30

(در فصل اول کتاب آرکین)

Arfken edition 3

Arfken-weber edition 5

استراخانیش
تدریس می شد

برجع :

8.5 نوبه میایتم

8.5 نوبه میایتم

3 نوبه تدریس

فصل 1

تحلیل برداری

1.1 تعریف برداریات

1.2 تعریف فضای جامع

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1.4 ضرب برداری

1.5 ضرب برداری

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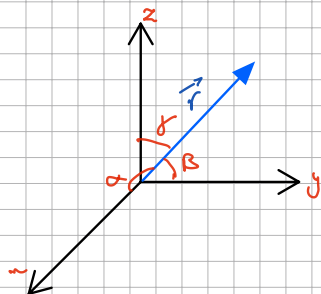
1.12 قضیه

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1.1. Definitions, elementary approach



$$x = r \cos \alpha$$

$$y = r \cos \beta$$

$$z = r \cos \gamma$$

سیدیں کی دی α, β, γ

$$|\vec{r}|^2 = x^2 + y^2 + z^2$$

$$= r^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)$$

$$1 = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$$

$$\vec{r} = (x, y, z) \rightarrow \vec{r} = (r; \cos \alpha, \cos \beta, \cos \gamma)$$

یہی ازایہا در جدول بالابردن می آید

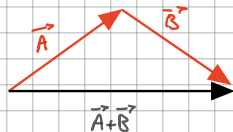
Unit Vectors. $\hat{i}, \hat{j}, \hat{k}$

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

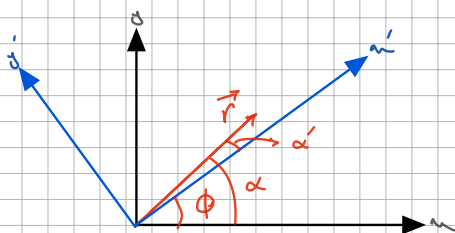
$A_x, A_y, A_z \rightarrow$ components

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\text{if } \vec{A} = 0 \leftrightarrow A_x = A_y = A_z = 0$$



1.2 Advanced differentiation



$$\vec{r} = x\hat{i} + y\hat{j} \rightarrow x = r \cos \alpha$$

$$y = r \sin \alpha$$

$$\vec{r} = x'\hat{i}' + y'\hat{j}' \rightarrow x' = r \cos \alpha'$$

$$y' = r \sin \alpha'$$

$$\rightarrow x' = x \cos \phi + y \sin \phi$$

$$y' = -x \sin \phi + y \cos \phi$$

$$x = x_1, x' = x'_1 \rightarrow a_{11} = \cos(\alpha'_1, \alpha_1) = \cos \phi$$

$$y = x_2, y' = x'_2 \rightarrow a_{12} = \cos(\alpha'_1, \alpha_2) = \sin \phi$$

$$a_{21} = \cos(\alpha'_2, \alpha_1) = -\sin \phi$$

$$a_{22} = \cos(\alpha'_2, \alpha_2) = \cos \phi$$

$$x'_1 = a_{11}x_1 + a_{12}x_2$$

$$x'_2 = a_{21}x_1 + a_{22}x_2$$

$$x'_i = \sum_{j=1}^2 a_{ij} x_j \quad (x_1, x_2, x_3, \dots, x_n) \xrightarrow[\text{matrix}]{\text{matrix}} (x'_1, x'_2, x'_3, \dots, x'_n)$$

$$\xrightarrow{\text{matrix}} x'_i = \sum_{j=1}^n a_{ij} x_j$$

$$a_{ij} = \cos(\alpha'_i, \alpha_j)$$

j = dummy index $\rightarrow$ ایس دی

i = free index $\rightarrow$ آزاد

جوتے ہر دو سبب

$$\rightarrow \begin{aligned} x_1 &= a_{11}x'_1 + a_{21}x'_2 \\ x_2 &= a_{12}x'_1 + a_{22}x'_2 \end{aligned}$$

$$a_{ij} = \frac{\partial x'_i}{\partial x_j} \quad \text{تبدیل کے سبب} \quad \frac{\partial x_j}{\partial x'_i}$$

$$A'_i = \sum_{j=1}^N \frac{\partial x'_i}{\partial x_j} A_j \stackrel{\text{تبدیل کے سبب}}{=} \sum_{j=1}^N \frac{\partial x_j}{\partial x'_i} A_j, \quad i \in \{1, 2, 3, \dots, N\}$$

شرایط تعامد

$$\alpha) \sum_{i=1}^N a_{ij} a_{ik} = a_{jk} = \begin{cases} 1 & j=k \\ 0 & j \neq k \end{cases} = \delta_{jk} \rightarrow \text{دلتا کرونکر}$$

$$\beta) \sum_{i=1}^N a_{ji} a_{ki} = a_{jk} = \delta_{jk}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix}, (A^T)_{ij} = A_{ji}$$

$$\gamma) \sum_{i=1}^N (A^T)_{ji} (A)_{ik} = (A^T \cdot A)_{jk} = \mathbb{I}_{jk} = \delta_{jk}$$

$$\delta) A A^T = \mathbb{I} = A^T A$$

ماتریس متعامد

$$\alpha) \sum_{i=1}^N a_{ij} a_{ik} = \sum_{i=1}^N \frac{\partial x'_i}{\partial x_j} \frac{\partial x'_i}{\partial x_k} = \sum_i \frac{\partial x'_i}{\partial x_j} \frac{\partial x_k}{\partial x'_i} = \frac{\partial x_k}{\partial x_j}$$

$$\frac{\partial x_k}{\partial x_j} = \delta_{jk}$$

نیز به همین شکل (ب) اثبات

$$\delta) \sum_i A^T_{ji} A_{ik} = \sum_i a_{ij} \cdot a_{ik} = (\alpha)$$

Vector space

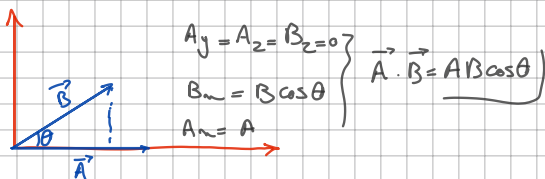
- 1) $\vec{x} = \vec{y} \quad m=1$ a) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
- 2) $\vec{x} + \vec{y} = \vec{z} \quad m+j_i = z_i$ b) $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
- 3) $a\vec{x} = (a x_1, a x_2, a x_3)$ c) $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$
- 3) $-\vec{x} = (-x_1, -x_2, -x_3)$ d) $(a+b)\vec{x} = a\vec{x} + b\vec{x}$
- 5) $\vec{0} = (0, 0, 0)$ e) $a(b\vec{x}) = (ab)\vec{x}$

3. $\text{Scalar product} = \text{Dot product} = \text{Inner product}$
 ضرب داخلی ضرب نقطه‌ای ضرب داخلی

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^N A_i B_i = A_x B_x + A_y B_y + A_z B_z = \vec{B} \cdot \vec{A} \quad \text{تغییر جایی}$$

تعریف هندسی: اگر دو بردار به صورت یکدیگر موازی نباشند، می‌توان آن‌ها را به دو بردار موازی و عمود بر هم تجزیه کرد.
 بقیه که از آن دو بردار موازی بگذرد

در آن صورت دستگاه مختصات را انتخاب می‌کنیم که یکی از بردار روی محور باشد



$\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{B}$ → یعنی با تغییر دستگاه حاصل تغییر نمی‌کند

$$\begin{aligned} \vec{A} \cdot \vec{B} &= \sum_{j=1}^N A_j B_j \\ \vec{A} \cdot \vec{B} &= \sum_{k=1}^N A'_k B'_k = \sum_{k=1}^N \left[\left(\sum_i a_{ki} A_i \right) \left(\sum_j b_{kj} B_j \right) \right] \\ &= \sum_i \sum_j \left[A_i B_j \left(\sum_k \underbrace{a_{ki} a_{kj}}_{\delta_{ij}} \right) \right] \\ &= \sum_i \sum_j (A_i B_j \delta_{ij}) = \sum_i A_i B_i = \vec{A} \cdot \vec{B} \end{aligned}$$

مثال $\vec{r} = (x, y, z, w)$ و $|\vec{r}| = 1 \Rightarrow x^2 + y^2 + z^2 + w^2 = 1$

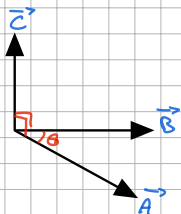
Find $\vec{t}$: $|\vec{t}| = 1$, $\vec{r} \cdot \vec{t}$

$$\begin{aligned} \text{حس} \cdot \vec{t} &= (y, -x, w, -z) \rightarrow t^2 = 1 \\ \vec{t} \cdot \vec{r} &= xy - yx + wz - wz = 0 \end{aligned} \quad \left. \begin{aligned} \text{حل} \cdot \vec{t} &= (x', y', z', w') \\ |\vec{t}| &= 1 \rightarrow x'^2 + y'^2 + z'^2 + w'^2 = 1 \end{aligned} \right\}$$

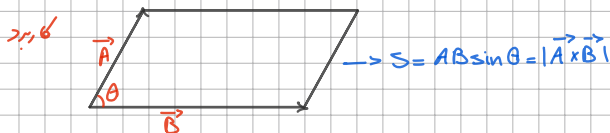
1.4 Vector or cross product

$$\vec{C} = \vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$C = AB \sin \theta$$



تقریب هندسی



$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k}$$

تقریب جبری

$$\vec{A} \cdot (\vec{A} \times \vec{B}) = A_x (\vec{A} \times \vec{B})_x + A_y (\vec{A} \times \vec{B})_y + A_z (\vec{A} \times \vec{B})_z = 0 \rightarrow \begin{matrix} \vec{A} \perp \vec{A} \times \vec{B} \\ \vec{B} \perp \vec{A} \times \vec{B} \end{matrix}$$

$$\text{مقدار: } (\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = A^2 B^2 \sin^2 \theta = |\vec{A} \times \vec{B}|^2$$

$$\left. \begin{matrix} A'_i = \sum_j a_{ij} A_j \\ B'_i = \sum_j a_{ij} B_j \end{matrix} \right\} \rightarrow C'_i = \sum_j a_{ij} C_j$$

$$\vec{C} = \vec{A} \times \vec{B}$$

$$C'_i = A'_j B'_k - A'_k B'_j \rightarrow C'_i = \left(\sum_l a_{jl} A_l \right) \left(\sum_m a_{km} B_m \right) - \left(\sum_l a_{kl} A_l \right) \left(\sum_m a_{jm} B_m \right)$$

$$i=3, j=1, k=2$$

$$= \sum_{l,m} (a_{jl} a_{km} - a_{kl} a_{jm}) A_l B_m$$

$$= \sum_{l,m} (a_{jl} a_{km} - a_{kl} a_{jm}) A_l B_m$$

معادلات:

$$\begin{cases} a_{11}a_{22} - a_{21}a_{12} = a_{33} \\ a_{13}a_{21} - a_{23}a_{11} = a_{32} \\ a_{12}a_{23} - a_{22}a_{13} = a_{31} \end{cases}$$

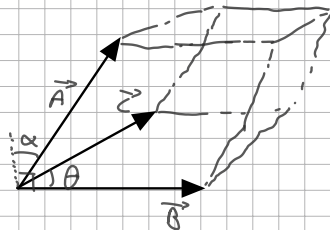
$$\rightarrow C'_3 = \begin{matrix} a_{33} & a_{32} & a_{31} \\ a_{33} & a_{32} & a_{31} \end{matrix} \left. \begin{matrix} A_1 B_2 + A_2 B_1 + A_3 B_3 \\ A_2 B_3 + A_3 B_2 + A_1 B_3 \end{matrix} \right\} \rightarrow C'_3 = \sum_{j=1}^3 a_{3j} C_j \rightarrow \text{همین ضرب برداری میسر است!}$$

$i=1, j=2, k=3$

15 Triple (scalar/cross) product

نکته ۱. $\vec{A} \cdot (\vec{B} \times \vec{C}) = A_x (\vec{B} \times \vec{C})_x + A_y (\vec{B} \times \vec{C})_y + A_z (\vec{B} \times \vec{C})_z$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = V_{ABC}$$

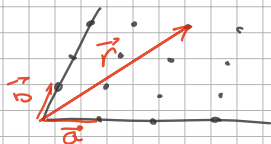


$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$V = \vec{A} \cdot (\vec{B} \times \vec{C}) = (\underbrace{BC \sin \theta}) \cdot (\underbrace{A \cos \alpha}) = V$$

حجم

reciprocal lattice متقابل شبکه



$$\vec{r} = n_a \vec{a} + n_b \vec{b} + n_c \vec{c}$$

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{\vec{a} \cdot (\vec{b} \times \vec{c})}$$

$$\vec{b}' = \frac{\vec{c} \times \vec{a}}{\vec{a} \cdot (\vec{b} \times \vec{c})}$$

$$\vec{c}' = \frac{\vec{a} \times \vec{b}}{\vec{a} \cdot (\vec{b} \times \vec{c})}$$

$$\left. \begin{array}{l} \vec{a}' = \frac{\vec{b} \times \vec{c}}{\vec{a} \cdot (\vec{b} \times \vec{c})} \\ \vec{b}' = \frac{\vec{c} \times \vec{a}}{\vec{a} \cdot (\vec{b} \times \vec{c})} \\ \vec{c}' = \frac{\vec{a} \times \vec{b}}{\vec{a} \cdot (\vec{b} \times \vec{c})} \end{array} \right\} \rightarrow a' a = b' b = c' c = 1$$

$$a' b = a' c = b' a = b' c = c' a = c' b = 0$$

یک شبکه دایره با دو متوجه

cross: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})$

یک باب

1.6. Gradient

* اگر $\varphi(x, y, z)$ یک اسکالر نقطه ای باشد، یعنی با تغییر مکان مقدار آن نیز تغییر کند.

$$\varphi'(x', y', z') = \varphi(x, y, z)$$

$$1) \frac{\partial \varphi'}{\partial x'_i} = \frac{\partial \varphi(x, y, z)}{\partial x'_i} \xrightarrow{\text{قانون زنجیره ای}} \sum_j \frac{\partial \varphi}{\partial x_j} \frac{\partial x_j}{\partial x'_i}$$

$$2) \frac{\partial \varphi'}{\partial x'_i} = \sum_j a_{ij} \frac{\partial \varphi}{\partial x_j} \xrightarrow{\text{ماتریس مؤلفه ای یک بردار}}$$

$$3) \vec{\nabla} \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right)$$

$$\downarrow$$

$$4) \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

example 1.6.1. $f(r) = f(\sqrt{x^2 + y^2 + z^2})$

$$\vec{\nabla} f(r) = \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z}$$

$$= \frac{df}{dr} \left(\hat{i} \frac{\cancel{x}}{\cancel{\partial x}} + \hat{j} \frac{\cancel{y}}{\cancel{\partial y}} + \hat{k} \frac{\cancel{z}}{\cancel{\partial z}} \right)$$

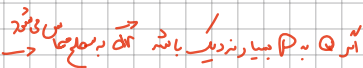
$$= \frac{df}{dr} \cdot \left(\frac{\cancel{\hat{i}x} + \cancel{\hat{j}y} + \cancel{\hat{k}z}}{r} \right) = f' \cdot \hat{r}$$

6

(دو به درفضا R^3) $\varphi(x, y, z) = C$: سطح همجدا

$\sqrt{c}$

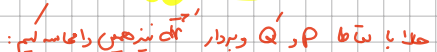
یہ



$$= d\psi(x, y, z)$$

$$\varphi = c \Rightarrow \vec{\nabla} \varphi \cdot \vec{dr} = 0 = \varphi(q) - \varphi(p)$$

$$\vec{\nabla} \varphi \perp d\vec{r}$$



$$\vec{\nabla} \varphi \perp d\vec{r}'$$

$$\vec{\nabla} \psi \perp (\psi = c \text{ سطح})$$

$$\cos \theta = 1 \Rightarrow (\vec{dr})_{\min}$$

$\theta = 0 \Rightarrow$

$$\Delta C = |\vec{\nabla} \cdot \varphi| |\vec{dr}| \cos \theta$$

$$dr = \frac{dc}{|\vec{\nabla} \varphi| \cos \theta}$$

$$(\Delta C)_{\text{max}} = |\nabla \varphi| (d\vec{r}_{\text{max}})$$

$\vec{\nabla} \varphi$ در جهت ماکسیم تغییرات فضای φ است

$$E \text{ 1.6.2: } \varphi(x, y, z) = (x^2 + y^2 + z^2)^{1/2} = r_c = C_i$$

مثال:

$$\vec{\nabla} \varphi(x, y, z) = \frac{x}{r} \hat{i} + \frac{y}{r} \hat{j} + \frac{z}{r} \hat{k} = \frac{\vec{r}}{r} = \frac{\vec{r}}{r_0} = \hat{r}$$

7 Divergence $\vec{\nabla} \cdot$

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \sum_{i=1}^3 \frac{\partial V_i}{\partial x_i}$$

$$* \vec{V} \cdot \vec{\nabla} = V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y} + V_z \frac{\partial}{\partial z} = \text{یک عدد اسکالر}$$

$$\rightarrow \vec{\nabla} \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \quad !!!$$

$$\vec{V} = \vec{r} \cdot f(r) \rightarrow \vec{\nabla} \cdot \vec{V} = \frac{\partial}{\partial x} [x f(r)] + \frac{\partial}{\partial y} [y f(r)] + \frac{\partial}{\partial z} [z f(r)]$$

$$= 3f(r) + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z}$$

$$= 3f(r) + \vec{r} \cdot \vec{\nabla} \cdot f(r) = 3f(r) + x \frac{df}{dr} \frac{\partial r}{\partial x} + y \frac{df}{dr} \frac{\partial r}{\partial y} + z \frac{df}{dr} \frac{\partial r}{\partial z}$$

$$= 3f(r) + r \frac{df}{dr}$$

$$\downarrow$$

$$\frac{x^2 + y^2 + z^2}{r} = \frac{r^2}{r} = r$$

تعبیر فیزیکی دیورتانس: یک سیال تراکم نپذیرد انستیه $\rho(x, y, z; t)$

در زمان t در نقطه x, y, z سرعت $\vec{v}$ صورت $\vec{V}(x, y, z; t)$

$$\rightarrow \vec{\nabla} \cdot (\rho \vec{V}) + \frac{\partial \rho}{\partial t} = 0 \rightarrow \text{معادله پیوستگی}$$

$\vec{\nabla} \cdot (\rho \vec{V})$: بهش دانسته در ادامه

$$\left. \begin{array}{l} x = ct \\ v = c \end{array} \right\} \rightarrow 0 = \vec{\nabla} \cdot (\rho \vec{V}) + \frac{\partial \rho}{\partial t} = \frac{\partial (\rho v^1)}{\partial x^1} + \frac{\partial (\rho v^2)}{\partial x^2} + \frac{\partial (\rho v^3)}{\partial x^3} + \frac{\partial (\rho v^0)}{\partial x^0}$$

$$= \sum_{\mu=0}^3 \frac{\partial (\rho v^\mu)}{\partial x^\mu} = 0$$

1.8 Curl $\vec{\nabla} \times$.

$$\vec{\nabla} \times \vec{V} = \hat{i} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{j} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{k} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)$$

$\vec{\nabla} \times \vec{V}$: یک عکس بردار

$$\vec{\nabla} \times [f(x, y, z) \cdot \vec{V}(x, y, z)] \Big|_m \text{ (مختصات)} = \frac{\partial f}{\partial y} \cdot v_z + \frac{\partial v_z}{\partial y} f - \frac{\partial f}{\partial z} v_y - \frac{\partial v_y}{\partial z} f$$

$$+ f \left[\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right] + \left[\frac{\partial f}{\partial y} v_z - \frac{\partial f}{\partial z} v_y \right]$$

$$= f \cdot \vec{\nabla} \times \vec{V} \Big|_m + (\vec{\nabla} f) \times \vec{V} \Big|_m$$

$$\vec{\nabla} \times (f \cdot \vec{V}) \stackrel{\text{قانون همبستگی}}{=} \vec{\nabla} \times (f \cdot \vec{V}) = f (\vec{\nabla} \times \vec{V}) + (\vec{\nabla} f) \times \vec{V}$$

$$\vec{\nabla} \cdot (f \cdot \vec{V}) = \left[\frac{\partial f}{\partial x} v_x + \frac{\partial f}{\partial y} v_y + \frac{\partial f}{\partial z} v_z \right] + \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right] f$$

$$= (\vec{\nabla} f) \cdot \vec{V} + f (\vec{\nabla} \cdot \vec{V})$$

if $\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \vec{B}$ is solenoidal (سولنوئید)

if $\vec{\nabla} \times \vec{V} = 0 \rightarrow \vec{V}$ is irrotational (غیر چرخشی)

Example 1.8.1.

$$\vec{\nabla} \times [\vec{r} f(r)] \Big|_m = \frac{\partial}{\partial y} [z f(r)] - \frac{\partial}{\partial z} [y f(r)]$$

$$= z \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial z}$$

$$= z \frac{df}{dr} \frac{\partial r}{\partial y} - y \frac{df}{dr} \frac{\partial r}{\partial z} = \frac{df}{dr} \cdot \left(\frac{yz}{r} - \frac{yz}{r} \right) = 0 \rightarrow \text{غیر چرخشی}$$

$$\vec{F}_G = -G \frac{mm}{r^3} \vec{r}$$

$$\vec{F}_C = k \frac{qQ}{r^3} \vec{r}$$

$$\vec{V} = \frac{C}{R} \vec{r}$$

$$\phi(r) = \frac{C}{r^3} \vec{r}$$

$$\nabla \times \vec{V} = 0$$

$$\vec{A} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}_B (\vec{A} \cdot \vec{B}) - (\vec{\nabla} \cdot \vec{A}) \vec{B}$$

یعنی یہاں نقطہ دیکھیں اس کی طرف

$$\vec{B} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}_A (\vec{A} \cdot \vec{B}) - (\vec{\nabla} \cdot \vec{B}) \vec{A}$$

$$\vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{A} \cdot \vec{B}) - \vec{A} \vec{\nabla} \cdot \vec{B} - \vec{B} \vec{\nabla} \cdot \vec{A}$$

$$\vec{\nabla} (\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + \vec{A} \vec{\nabla} \cdot \vec{B} + \vec{B} \vec{\nabla} \cdot \vec{A}$$

مسلم میزبانی اور صرف

19. Successive application ∇

$$a) \vec{\nabla} \vec{\nabla} \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \text{Laplacian of } \varphi = \nabla^2 \varphi$$

$\nabla^2 \varphi$: φ پائسل (الترامستیک) باشد :

Example 19.1. $\nabla^2 g(r) = ?$

$$\text{Ex 19.1} \Rightarrow \vec{\nabla}(\vec{r} \cdot f(r)) = 3f(r) + r \frac{df(r)}{dr}$$

$$\begin{aligned} \vec{\nabla} \vec{\nabla} g(r) &= \vec{\nabla} \left(\hat{r} \cdot \frac{dg(r)}{dr} \right) = \vec{\nabla} \cdot \left(\vec{r} \cdot \frac{1}{r} \cdot \frac{dg(r)}{dr} \right) \\ &= \frac{3}{r} \frac{dg(r)}{dr} + r \cdot \frac{d}{dr} \left[\frac{1}{r} \frac{dg}{dr} \right] \\ &= \frac{2}{r} \frac{dg}{dr} + \frac{d^2}{dr^2} g(r) \end{aligned}$$

$$b) \vec{\nabla} \times \vec{\nabla} \varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix} = \begin{pmatrix} \frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y} \\ \dots \\ \dots \end{pmatrix} \hat{i} + \dots \hat{j} + \dots \hat{k} = \vec{0}$$

$\vec{\nabla} \times \vec{\nabla} \varphi$ غیر صفری نیست! (یعنی بردار)

$$c) \vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = 0$$

بسی $\vec{\nabla} \times \vec{v}$ یک سکلر نیست!

$$d) \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) = \vec{\nabla} \varphi$$

$$\varphi = \vec{\nabla} \cdot \vec{v}$$

$$e) \vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - (\vec{\nabla} \cdot \vec{\nabla}) \vec{v} = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \nabla^2 v_x \hat{i} - \nabla^2 v_y \hat{j} - \nabla^2 v_z \hat{k}$$

تجسید از معادله 2 در آن ∇^2 لاپلاسین (تدریجی شده است)

تدریجی از معادله 2 در آن ∇^2 لاپلاسین (تدریجی شده است)

Example 1.9.2

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \xrightarrow{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} \quad (-\vec{\nabla} \times \vec{E})$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \rightarrow \vec{E} \text{ معادله موج}$$

D'Alembertian $\square = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \leadsto \square \vec{E} = 0$

in the (m, n) -dimension

$$\square = \sum_{i=1}^m \frac{\partial^2}{\partial x_i^2} - \frac{1}{c^2} \sum_{j=1}^n \frac{\partial^2}{\partial t_j^2}$$

10. vector integration

انتقال خطی: $\int_C \varphi(x, y, z) \cdot d\vec{r} = \hat{i} \int_C \varphi \, dx + \hat{j} \int_C \varphi \, dy + \hat{k} \int_C \varphi \, dz$

که یک معنی درفضا است که ممکن است بازتاب داشته باشد

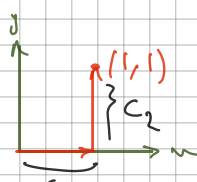
$$\begin{matrix} x = x(y) & y = x(y) & x = x(z) \\ z = z(y) & z = z(y) & y = y(z) \end{matrix}$$

* باید مسیر C داده شده باشد تا x, y, z را بتوانی بر حسب یکدیگر بنویسی!

$$W = \int_C \vec{F}(x, y, z) \cdot d\vec{r} = \int_C F_x(x, y, z) \, dx + \int_C F_y(x, y, z) \, dy + \int_C F_z(x, y, z) \, dz$$

اگر $\vec{F}$ نیرو باشد حاصل کار نیرو می شود

Example 1.10.1: $\vec{F} = -y\hat{i} + x\hat{j}$: $\int_C \vec{F} \cdot d\vec{r} = ?$



$$\int_{(0,0)}^{(1,1)} (-y\hat{i} + x\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) =$$

$$\int_{C_1} (-y \, dx + x \, dy) + \int_{C_2} (-y \, dx + x \, dy) = 1$$

0 1

$$\int_C \vec{v}(x, y, z) \times d\vec{r}$$

- استرال سطحی

اثر سطح بسته باشد جهت $\vec{d\vec{\sigma}}$ به سمت بیرون و محدود به سطح است

اگر بسته نباشد = از قانون دست راست



1- $\int_S \varphi(x, y, z) d\vec{\sigma}$

2- $\int_S \vec{V}(x, y, z) \cdot d\vec{\sigma} =$

نظری است که بر اساس میدان برداری
از سطحی عبور می کند

3- $\int_S \vec{V}(x, y, z) \cdot d\vec{\sigma}$

انتگرال حجم -

$$\int_V \varphi(x, y, z) \cdot dV$$

$$\int_V \vec{w}(x, y, z) \cdot dV$$

تعاریف گزینش و دیفرانسیل

به صورت انتگرالی

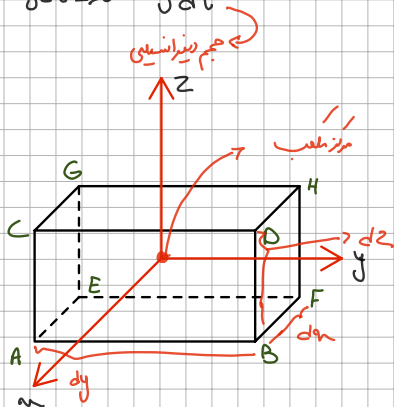
$$\vec{\nabla} \varphi = \lim_{\int dV \rightarrow 0} \frac{\int \varphi d\vec{\sigma}}{\int dV}$$

$$\vec{\nabla} \cdot \vec{v} = \lim_{\int dV \rightarrow 0} \frac{\int \vec{v} \cdot d\vec{\sigma}}{\int dV}$$

$$\vec{\nabla} \times \vec{v} = \lim_{\int dV \rightarrow 0} \frac{\int d\vec{\sigma} \times \vec{v}}{\int dV}$$

$$\int_{x_0}^{x_0+dx} dx = \int_{x_0}^{x_0+dx} dx \int_{y_0}^{y_0+dy} dy \int_{z_0}^{z_0+dz} dz = dx dy dz$$

$$\vec{\nabla} \varphi = \lim_{\Delta \tau \rightarrow 0} \frac{\int \varphi d\vec{\sigma}}{\Delta \tau}$$



$$\int_{EFGH} \varphi d\vec{\sigma} = \int_{EFGH} \varphi \left(-\frac{dx}{2}, y, z\right) dy dz \hat{i}$$

$$= -\hat{i} \int \left[\varphi(x, y, z) + \frac{\partial \varphi}{\partial x} \left(-\frac{dx}{2}\right) \right] dy dz$$

$$\int_{ABCD} \varphi d\vec{\sigma} = \int_{ABCD} \varphi \left(\frac{dx}{2}, y, z\right) (\hat{i} dy dz)$$

$$= \hat{i} \int \left[\varphi(x, y, z) + \frac{\partial \varphi}{\partial x} \cdot \frac{dx}{2} \right] dy dz$$

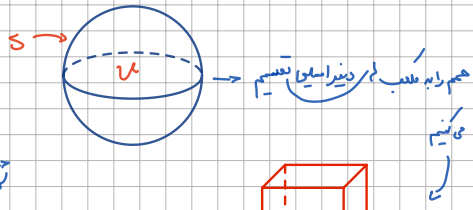
$$\Rightarrow \int_{ABCD} + \int_{EFGH} = \hat{i} \int \frac{\partial \varphi}{\partial x} dx dy dz = \hat{i} \frac{\partial \varphi}{\partial x} dx dy dz$$

$$\rightarrow \int \varphi d\vec{\sigma} = \left[\hat{i} \frac{\partial \varphi}{\partial x} + \hat{j} \frac{\partial \varphi}{\partial y} + \hat{k} \frac{\partial \varphi}{\partial z} \right] dx dy dz = \vec{\nabla} \cdot \varphi \cdot d\tau$$

$$\rightarrow \lim_{\Delta \tau} \frac{\int \varphi d\vec{\sigma}}{\Delta \tau} = \vec{\nabla} \varphi \checkmark$$

1.11. The Gauss's theorem

$$\int_V \vec{\nabla} \cdot \vec{v} \, d\tau = \int_S \vec{v} \cdot d\vec{\sigma} \rightarrow \text{proof:}$$



$$S = \partial V$$

در هم می آید

$$1.61. \int_V \vec{\nabla} \cdot \vec{v} \, d\tau = \int_V (\vec{\nabla} \cdot (\rho \vec{v})) \, d\tau$$

$$\sum \vec{v} \cdot d\vec{\sigma} = \vec{\nabla} \cdot \vec{v} \, d\tau : \rho \vec{v} = \vec{v}$$

$$\sum_{\text{تمام مکعب}} \sum_{\text{هر یک از مکعب}} \vec{v} \cdot d\vec{\sigma} = \sum_{\text{تمام مکعب}} \vec{\nabla} \cdot \vec{v} \, d\tau$$

تشریح: اگر مکعب را به هم چسبده از یکدیگر جدا کنیم، جمع می کنیم و می بینیم که شارهای بیرونی از یکدیگر وسط می آید، خط می خیزد، پس داریم

$$\sum_{\text{تمام مکعب}} \vec{v} \cdot d\vec{\sigma} = \sum_{\text{تمام مکعب}} \vec{\nabla} \cdot \vec{v} \, d\tau$$

اصلاح کردی

$$\oint_S \vec{v} \cdot d\vec{\sigma} = \oint_V \vec{\nabla} \cdot \vec{v} \, d\tau$$

The Green's theorem

$$u(x, y, z) \quad ① \vec{\nabla} (u \vec{\nabla} v) = \vec{\nabla} u \vec{\nabla} v + u \nabla^2 v$$

$$v(x, y, z) \quad ② \vec{\nabla} (v \vec{\nabla} u) = \vec{\nabla} v \vec{\nabla} u + v \nabla^2 u$$

$$\rightarrow \vec{\nabla} (u \vec{\nabla} v - v \vec{\nabla} u) = u \nabla^2 v - v \nabla^2 u$$

$$\xrightarrow[\text{مجموع}]{\text{استدلال}} \int_{\mathcal{U}} \vec{\nabla} (u \vec{\nabla} v - v \vec{\nabla} u) d\vec{\tau} = \int_{\mathcal{U}} (u \nabla^2 v - v \nabla^2 u) d\tau$$

$$\Rightarrow \int_{\mathcal{S}} (u \vec{\nabla} v - v \vec{\nabla} u) d\vec{\sigma} = \int_{\mathcal{U}} (u \nabla^2 v - v \nabla^2 u) d\tau \rightarrow \text{فصل اول قضیه گرین}$$

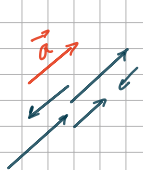
$$\xrightarrow{\text{بند اول}} \int_{\mathcal{U}} \vec{\nabla} (u \vec{\nabla} v) d\tau = \int_{\mathcal{U}} (\vec{\nabla} u \vec{\nabla} v + u \nabla^2 v) d\tau$$

$$\Rightarrow \oint_{\mathcal{S}} (u \vec{\nabla} v) \cdot d\vec{\sigma} = \int_{\mathcal{U}} (\vec{\nabla} u \vec{\nabla} v + u \nabla^2 v) d\tau \rightarrow$$

فصل دوم قضیه گرین

- Alternative forms of the Gauss's theorem

1) $\vec{V}(m, y, z) = \varphi(m, y, z) \vec{a} \rightarrow$ بجانب



$$\int_V \vec{\nabla} \cdot (\vec{a} \varphi) d\tau = \int_S \vec{a} \varphi d\vec{\sigma}$$

$$\# \vec{\nabla}(\varphi \vec{a}) = \vec{\nabla} \varphi \vec{a} + \varphi \vec{\nabla} \vec{a}$$

$$\int_V \vec{a} \vec{\nabla} \varphi d\tau = \vec{a} \int_S \varphi d\vec{\sigma}$$

$$\vec{a} \cdot \left[\int_V \vec{\nabla} \varphi d\tau - \int_S \varphi d\vec{\sigma} \right] = 0 \rightarrow \int_V \vec{\nabla} \varphi d\tau = \int_S \varphi d\vec{\sigma}$$

2) $\vec{V}(m, y, z) = \vec{a} \times \vec{P}(m, y, z)$ بجانب

$$\int_V \vec{\nabla} \cdot (\vec{a} \times \vec{P}) d\tau = \int_S (\vec{a} \times \vec{P}) \cdot d\vec{\sigma}$$

$$\int_V \vec{a} \cdot (-\vec{\nabla} \times \vec{P}) d\tau = \int_S \vec{a} \cdot (\vec{P} \times d\vec{\sigma})$$

$$\int_V \vec{\nabla} \times \vec{P} d\tau = \int_S d\vec{\sigma} \times \vec{P}$$

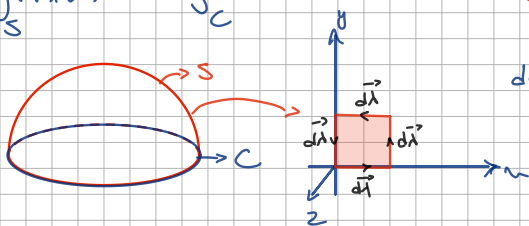
1.12 The Stokes's theorem

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{\sigma} = \oint_C \vec{v} \cdot d\vec{\lambda}$$

سطح S رو به مربع S در فضای 3D تقسیم می کنیم

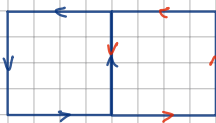
$$d\vec{\sigma} = \hat{k} \, dx \, dy$$

$$(\vec{\nabla} \times \vec{v})_z \, dx \, dy$$



$$\sum_{\text{هر چهار ضلع}} \vec{v} \cdot d\vec{\lambda} = (\vec{\nabla} \times \vec{v}) \cdot d\vec{\sigma}$$

برای هر یک از ضلعها



در هر دو ضلع که جهت $d\vec{\lambda}$ مخالف هم می باشد

$$\vec{v} \cdot d\vec{\lambda}|_1 + \vec{v} \cdot d\vec{\lambda}|_2 = 0$$

$$\sum_{\text{هر چهار ضلع}} \sum \vec{v} \cdot d\vec{\lambda} = \sum (\vec{\nabla} \times \vec{v}) \cdot d\vec{\sigma} = \oint_C \vec{v} \cdot d\vec{\lambda} = \int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{\sigma}$$

Alternative forms of the Stokes's theorem

$$1) \vec{v}(x, y, z) = \vec{a} \cdot \nabla \varphi(x, y, z) \Rightarrow \int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{\sigma} = \oint_C \vec{v} \cdot d\vec{\lambda} = \oint_C \varphi(x, y, z) \cdot \vec{a} \cdot d\vec{\lambda}$$

$$\hookrightarrow \int_S [(\vec{\nabla} \varphi) \times \vec{a}] \cdot d\vec{\sigma} = \vec{a} \cdot \int_S \nabla \varphi \cdot d\vec{\sigma} \Rightarrow \vec{a} \cdot \left[\int_S d\vec{\sigma} \times \nabla \varphi - \oint_C \varphi \, d\vec{\lambda} \right] = 0$$

$$\Rightarrow \int_S d\vec{\sigma} \times \nabla \varphi = \oint_C \varphi \cdot d\vec{\lambda}$$

نیم دوم تقسیم می کنیم

$$2) \vec{v} = \vec{a} \times \vec{P}(x, y, z) \Rightarrow \int_S (d\vec{\sigma} \times \vec{v}) \cdot \vec{P} = \oint_C d\vec{\lambda} \cdot \vec{P}$$

نیم سوم

⊛ Combination of the Gauss's and Stokes's theorems:

$$\int_V \vec{\nabla} \cdot \vec{v} \, d\tau = \int_S \vec{v} \cdot d\vec{\sigma} \quad (\text{Gauss's}) \quad S = \partial V$$

$$\vec{v} = \vec{\nabla} \times \vec{w}$$

$$C = \partial S$$

$$\Rightarrow \int_V \underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{w})}_{=0} \, d\tau = \oint_S (\vec{\nabla} \times \vec{w}) \cdot d\vec{\sigma} = \oint_C \vec{w} \cdot d\vec{\lambda} = 0 \xrightarrow{\vec{w}=0} C = \partial S = \overset{\text{closed}}{\partial} V = 0$$

13. potential theorem

scalar potential

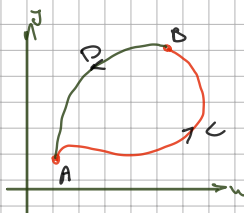
$$\vec{F} = -\vec{\nabla}\varphi \rightarrow \varphi \text{ پتانسیل اسکالر می نامیم}$$

$$\vec{\nabla} \times \vec{F} = -\vec{\nabla} \times (\vec{\nabla} \varphi) = 0$$

$\vec{F}$ غیر چرخشی یا پتانسیل دارد!

$$\textcircled{2} \vec{F} \cdot d\vec{r} = -\vec{\nabla}\varphi \cdot d\vec{r}$$

$$\oint_C \vec{F} \cdot d\vec{r} = -\oint_C d\varphi = 0$$



$$W_{ACBDA} = \int \vec{F} \cdot d\vec{r} = 0$$

$$\int_{ACB} \vec{F} \cdot d\vec{r} + \int_{BDA} \vec{F} \cdot d\vec{r} = 0$$

$$-\int_{ADB} \vec{F} \cdot d\vec{r}$$

$$W_{ACB} = W_{ADB} \rightarrow \text{کمیته پتانسیل را می بیند از مسیر}$$

درست آوردن $\vec{F} = -\vec{\nabla}\varphi$ از معادله کار

$$\int_A^B \vec{F} \cdot d\vec{r} = -\int_A^B d\varphi = \varphi(A) - \varphi(B)$$

$$\Rightarrow \int_A^B \vec{F} \cdot d\vec{r} = \varphi(A) - \varphi(B) \xrightarrow{d} \vec{F} \cdot d\vec{r} = -\vec{\nabla}\varphi \cdot d\vec{r}$$

$$(\vec{F} + \vec{\nabla}\varphi) \cdot d\vec{r} = 0$$

$$\vec{F} = -\vec{\nabla}\varphi$$

پتانسیل دارد

$$(\vec{F} = -\vec{\nabla}\varphi) \Rightarrow (\vec{\nabla} \times \vec{F} = 0) \Rightarrow \left(\oint_C \vec{F} \cdot d\vec{r} = 0 \right)$$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma} = 0 \Rightarrow (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma} = 0$$

$$\vec{\nabla} \times \vec{F} = 0$$

درست آوردن $\vec{\nabla} \times \vec{F} = 0$ از $\oint_C \vec{F} \cdot d\vec{r} = 0$

Example 1.13.1 Gravitational potential

$$\vec{F} = -G \frac{Mm}{r^2} \vec{r}_0 = -\frac{k \vec{r}_0}{r^2}$$

$$\vec{\nabla}_x (\vec{r} f(r)) = 0, f(r) = \frac{k}{r^3}$$

$$\vec{\nabla}_x \vec{F} = 0 \leadsto \vec{F} = -\vec{\nabla} \varphi \Rightarrow \vec{F} d\vec{r} = -d\varphi$$

$$\vec{r}_0 \cdot d\vec{r} = dr$$

$$\int_r^\infty \vec{F} \cdot d\vec{r} = \varphi(\infty) - \varphi(r)$$

$$-k \int_0^\infty \frac{\vec{r}_0 \cdot d\vec{r}}{r^2} = -\varphi(r), \varphi(\infty) = 0$$

Example 1.13.2 Centrifugal potential

$$\vec{F} = \omega^2 r \vec{r}_0$$

$$\Rightarrow \vec{\nabla}_x \vec{F} = 0 \Rightarrow F = -\vec{\nabla} \varphi$$

$$\int_r^\infty \vec{F} \cdot d\vec{r} = \int_0^r -d\varphi = \varphi(\infty) - \varphi(r)$$

$$\omega^2 \int_0^r r \cdot \vec{r}_0 \cdot d\vec{r} = -\varphi(r) \rightarrow \varphi(r) = -\frac{1}{2} \omega^2 r^2 = -\frac{1}{2} \omega^2 (x^2 + y^2 + z^2)$$

$$\vec{F} = -\frac{Gm_1 m_2 \vec{r}_0}{r^2} \propto r^n, 1 \leq n \leq N \leq 9$$

$\alpha \ll 1$

Exact differentiation

$$P(x,y) dx + Q(x,y) dy = df(x,y)$$

$$\left. \begin{aligned} P &= \frac{\partial f}{\partial x} \\ Q &= \frac{\partial f}{\partial y} \end{aligned} \right\} \rightarrow \left. \begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial Q}{\partial x} &= \frac{\partial^2 f}{\partial y \partial x} \end{aligned} \right\} \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0 \quad (*)$$

$$\vec{\nabla}_x \vec{F} = 0 \Rightarrow \vec{\nabla} F/2 = 0 \leadsto \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$$

$$P(x,y,z) dx + Q(x,y,z) dy + R(x,y,z) dz$$

تحت چه شرایطی عبارت زیر یک دیفرانسیل کامل است؟

or

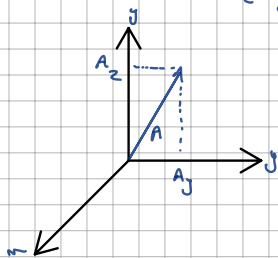
$$\sum_{i=1}^n f_i(x_1, x_2, \dots, x_n) dx_i$$

⊗ Vector potential:

$$\vec{\nabla} \cdot \vec{B} = 0 \leadsto \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{B} = \hat{i} B_x + \hat{j} B_y + \hat{k} B_z \quad \left\{ \begin{array}{l} B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{array} \right.$$

$$\vec{A} = \hat{i} A_x + \hat{j} A_y + \hat{k} A_z$$



حسبانه مختصات انتخاب می کنیم $A_x = 0$

$$A_x = 0 \leadsto \left\{ \begin{array}{l} B_y = -\frac{\partial A_z}{\partial x} \leadsto A_z = -\int B_y dx + f_3(y,z) \\ B_z = \frac{\partial A_y}{\partial x} \leadsto A_y = \int B_z dx + f_2(y,z) \end{array} \right.$$

$$B_x(x,y,z) = \left(-\int_{x_0}^x \frac{\partial B_x}{\partial y} dx + \frac{\partial f_3}{\partial y} \right) - \left(\int_{x_0}^x \frac{\partial B_z}{\partial z} dx + \frac{\partial f_2}{\partial z} \right)$$

$$= \int_{x_0}^x \underbrace{\left(-\left(\frac{\partial B_y}{\partial y} + \frac{\partial B_y}{\partial z} \right) \right)}_{\frac{\partial B_x}{\partial x}} dx + \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) = B_x = B_x \Big|_{x_0}^x + \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}$$

$$\Rightarrow B_x(x_0, y, z) = \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}$$

$$0 = f_3 - \text{w.t.t} \leadsto f_3 = \int_{y_0}^y B_x(x_0, y, z) dy$$

$$\vec{A} = \hat{j} \int_{x_0}^x B_z(x, y, z) dx + \hat{k} \int_{y_0}^y B_x(x_0, y, z) dy$$

$$\vec{B} = \hat{i} b_1 + \hat{j} b_2 + \hat{k} b_3$$

$$\vec{A} = \hat{i} a_1 + \hat{j} a_2 + \hat{k} a_3 \quad \xrightarrow{\text{المساواة}} a_1 = 0$$

$$\vec{A}' = \vec{A} + \vec{\nabla} \psi$$

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times \vec{\nabla} \psi}_0$$

$$= \vec{\nabla} \times \vec{A} = \vec{B}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$$

2.14 Gauss's Law, Poisson's equation

$$\vec{E} = \frac{q \vec{r}_0}{4\pi\epsilon_0 r^2} \rightarrow \oint_S \vec{E} \cdot d\vec{\sigma} = \begin{cases} \frac{q}{\epsilon_0} & \text{اگر } q \text{ شامل بار باشد} \\ 0 & \text{اگر } q \text{ شامل بار نباشد} \end{cases}$$

$$\oint_S \vec{v} \cdot d\vec{\sigma} = \int_V \vec{\nabla} \cdot \vec{v} \cdot d\tau$$

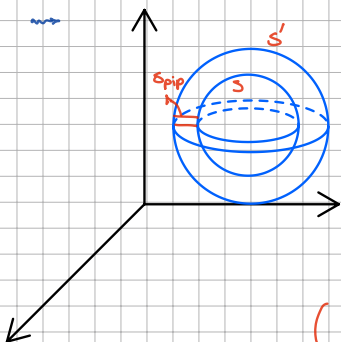
$$S = \partial V \leadsto \vec{r} = \frac{\vec{r}_0}{r^2} \rightarrow \oint_S \frac{\vec{r}_0 \cdot d\vec{\sigma}}{r^2} = \int_V \vec{\nabla} \cdot \frac{\vec{r}_0}{r^2} d\tau$$

$$\vec{\nabla}(\vec{r} f(r)) = 3f(r)\vec{r} + r \frac{df}{dr}$$

$$f(r) = r^{n-1} \leadsto \vec{\nabla} \cdot (\vec{r}_0 r^{n-1}) = 3r^{n-1} + (n-1)r^{n-1} = (n+2)r^{n-1}$$

$$n = -2 \leadsto \vec{\nabla} \cdot \left[\frac{\vec{r}_0}{r^2} \right] = 0$$

$$\oint_S \frac{\vec{r}_0 \cdot d\vec{\sigma}}{r^2} = 0 \xrightarrow{n=-2} \oint_S \vec{E} \cdot d\vec{\sigma} = 0$$



$$\oint_{S_{\text{Tot}}} \vec{E} \cdot d\vec{\sigma} = 0$$

$$S_{\text{Tot}} = S + S' + S_{\text{pp}}$$

$$\left(\int_S + \int_{S'} + \int_{S_{\text{pp}}} \right) \vec{E} \cdot d\vec{\sigma} = 0$$

$$\int_{S_{\text{pp}}} \vec{E} \cdot d\vec{\sigma} \rightarrow 0$$

$$\int_S \frac{\vec{r}_0 \cdot d\vec{\sigma}}{r^2} + \int_{S'} \frac{\vec{r}_0 \cdot d\vec{\sigma}'}{\delta^2} = 0 \rightarrow d\vec{\sigma} = -\hat{r} \delta^2 d\Omega$$

شعاع رادیان

$$\hookrightarrow \int_S \frac{\hat{r} \cdot d\vec{\sigma}}{r^2} - \int_{S'} \frac{\hat{r} \cdot \hat{r} \delta^2 d\Omega}{\delta^2} = \int_{S'} d\Omega = 4\pi$$

$$\oint_S \vec{E} \cdot d\vec{\sigma} = \frac{q}{\epsilon_0} = \frac{\int_V \rho d\tau}{\epsilon_0} \rightarrow \int_V \vec{\nabla} \cdot \vec{E} \cdot d\tau = \int_V \frac{\rho}{\epsilon_0} d\tau \leadsto \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

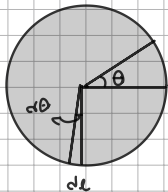
$$\vec{\nabla} \cdot \vec{E} \sim \frac{\partial \vec{B}}{\partial t} \xrightarrow{\text{قانون أمبير المعدل}} \vec{\nabla} \cdot \vec{E} = 0 \leadsto \vec{E} = -\vec{\nabla} \phi \leadsto \nabla^2 \phi = -\frac{\rho}{\epsilon_0} \rightarrow \text{معادلات بواسون}$$

$$\vec{E}_G = -G \frac{m_1 m_2}{r^2} \vec{r}_0 \Rightarrow \phi_G = -\frac{G m_1 m_2}{r}$$

$$\nabla^2 \phi = 4\pi G \rho_{\text{mass}}$$

$$S^2 \Rightarrow n_1^2 + n_2^2 = R^2$$

$$dl = R d\theta$$



$$S^n \Rightarrow \sum_{i=1}^{n+1} n_i^2 = R^2$$

$$\theta_1, \theta_2, \theta_3, \dots, \theta_n$$

$$d\Omega_n = ?$$

$$S^2 \Rightarrow n_1^2 + n_2^2 + n_3^2 = R^2$$

$$\theta, \varphi \rightarrow d\Omega = \sin \theta d\theta d\varphi$$

$$\int d\Omega = \int_0^{2\pi} \int_0^\pi \sin \theta d\theta d\varphi$$

1.15 Helmholtz's theorem

Uniqueness theorem قضیه یکتایی

Proof. $\vec{\nabla} \times \vec{V}_1 = \vec{C}$

موضوعی کنیم که V_2 نیز در دسترس یک معلوم V_1 صدق می کند

$$\vec{\nabla} \cdot \vec{V}_1 = s$$

$$\vec{w} = \vec{V}_1 - \vec{V}_2$$

$$V_{1n} = \lambda$$

$$\vec{\nabla} \cdot \vec{w} = \vec{\nabla} \cdot \vec{V}_1 - \vec{\nabla} \cdot \vec{V}_2 = 0 \quad (\alpha)$$

$$\vec{\nabla} \times \vec{w} = \vec{\nabla} \times \vec{V}_1 - \vec{\nabla} \times \vec{V}_2 = 0 \quad (\beta)$$

$$w_n = V_{1n} - V_{2n} = \lambda - \lambda = 0 \quad (\gamma)$$

$$\text{قضیه یکتایی} \Rightarrow \int_S u \vec{\nabla} v \cdot d\vec{\sigma} = \int_V \vec{\nabla} u \cdot \vec{\nabla} v d\tau + \int_U u \vec{\nabla}^2 v d\tau$$

$$u = v = \varphi \Rightarrow \int_S \varphi \vec{\nabla} \varphi \cdot d\vec{\sigma} = \int_V \vec{\nabla} \varphi \cdot \vec{\nabla} \varphi d\tau + \int_U \varphi \vec{\nabla}^2 \varphi d\tau$$

$$\ominus (\beta) \rightarrow \vec{\nabla}^2 \varphi = 0 \quad \vec{w} \cdot \vec{w}$$

$$-\int_S \varphi \vec{w} \cdot (\hat{n} dA) = \int_U |\vec{w}|^2 d\tau + \int_U \varphi \vec{\nabla}^2 \varphi d\tau$$

$$-\int_S \varphi \underbrace{w_n}_{=0} dA = \int_U |\vec{w}|^2 d\tau = 0 \rightsquigarrow \vec{w} = 0 \rightsquigarrow \vec{V}_1 = \vec{V}_2 \quad (\checkmark)$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$0 = \vec{\nabla} \cdot \vec{A}$$

$$A_n = \vec{A} \cdot \hat{n}$$

The Dirac delta-function

δ -function

$$\delta(x-a) = 0, x \neq a$$

$$\int_{(x-a) < \varepsilon} f(x) \delta(x-a) dx = f(a)$$

$$\int_{-\infty}^{+\infty} f(x) \delta(x-a) dx = \int_{a-\varepsilon}^{a+\varepsilon} f(x) \delta(x-a) dx$$

نقطه: δ

$$\delta(\vec{r}-\vec{a}) = 0, \vec{r} \neq \vec{a}$$

نقطه: δ

$$\int_V f(\vec{r}) \delta(\vec{r}-\vec{a}) d\tau = f(\vec{a})$$

$$\vec{a} = (a_1, a_2, a_3)$$

همه مختصات (a_1, a_2, a_3) را در نظر بگیرید

$$\nabla^2 g(r) = \frac{2}{r} \frac{dg}{dr} + \frac{d^2g}{dr^2}$$

$$g = r^n \Rightarrow \nabla^2 r^n = n(n+1) r^{n-2}$$

$$n = -1, r=0 \Rightarrow \nabla^2 \frac{1}{r} = 0$$

$$\left\{ \begin{aligned} \int_V \nabla^2 \frac{1}{r} d\tau &= \int_V \vec{\nabla} \cdot \vec{\nabla} \frac{1}{r} d\tau = - \oint_V \vec{\nabla} \frac{1}{r} \cdot \vec{n} d\sigma \\ &= \begin{cases} -4\pi & \text{اگر } \vec{r} \text{ در داخل } V \text{ باشد} \\ 0 & \text{اگر } \vec{r} \text{ خارج } V \text{ باشد} \end{cases} \Rightarrow -\frac{1}{4\pi} \int_V \nabla^2 \frac{1}{r} d\tau = 1 \end{aligned} \right.$$

$$\xrightarrow{\text{نزدیک } \vec{r}} \delta(\vec{r}) = \frac{1}{4\pi} \nabla^2 \frac{1}{|\vec{r}|} \Rightarrow \nabla^2 \frac{1}{|\vec{r}_1 - \vec{r}_2|} = -4\pi \delta(\vec{r}_1 - \vec{r}_2)$$

نقطه: δ در $\vec{r}_1$ عمل می کند

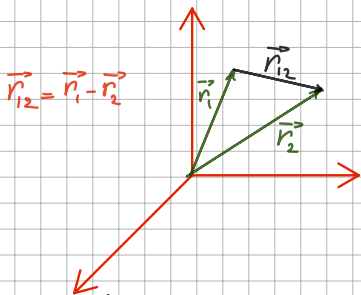
The Helmholtz's theorem

$$\vec{\nabla} \cdot \vec{V} = S(\vec{r}) \quad , \quad \vec{\nabla} \times \vec{V} = \vec{C}(\vec{r})$$

$$S(\infty) = 0 \quad \vec{C}(\infty) = 0$$

$$\vec{V} = -\vec{\nabla} \phi + \vec{\nabla} \times \vec{A} \rightarrow \begin{cases} \phi(\vec{r}_1) = \frac{1}{4\pi} \int \frac{S(\vec{r}_2) d\tau_2}{r_{12}} \\ \vec{A}(\vec{r}_1) = \frac{1}{4\pi} \int \frac{\vec{C}(\vec{r}_2) d\tau_2}{r_{12}} \end{cases}$$

اگر میدان برداری $V(r)$ در روابط زیر صدق کند
 آن سازهی توان V را به حسب یک قیمت غیر جبری
 و یک قیمت سولویبر پورت



$$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$

این V در معادلات $\Delta \phi = S$ و $\vec{\nabla} \times \vec{A} = \vec{C}$ صدق کند
 آن را در رابطه $\vec{V} = -\vec{\nabla} \phi + \vec{\nabla} \times \vec{A}$ به حسب S و $\vec{C}$ مربوط ϕ و $\vec{A}$ دارد
 بردار میانی

proof. $\nabla \cdot \vec{V} = -\nabla^2 \phi = S$
 $\vec{\nabla} \times \vec{V} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{C}$

$$\vec{\nabla}_1 \phi(\vec{r}_1) = \frac{1}{4\pi} \int S(\vec{r}_2) \vec{\nabla}_1 \frac{1}{r_{12}} d\tau_2 = \frac{1}{4\pi} \int S(\vec{r}_2) (-4\pi \delta(\vec{r}_1 - \vec{r}_2)) d\tau_2 = S(\vec{r}_1) \quad \checkmark$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$-\nabla^2 \vec{A} = \frac{1}{4\pi} \int \vec{C}(\vec{r}_2) \vec{\nabla}_1 \frac{1}{r_{12}} d\tau_2 = \int \vec{C}(\vec{r}_2) \delta(\vec{r}_1 - \vec{r}_2) d\tau_2 = \vec{C}(\vec{r}_1)$$

$$\vec{\nabla}_1 (\vec{\nabla}_1 \cdot \vec{A}(\vec{r}_1)) = \frac{1}{4\pi} \int \vec{\nabla}_1 \cdot \left(\vec{C}(\vec{r}_2) \vec{\nabla}_1 \frac{1}{r_{12}} \right) d\tau_2 = \frac{1}{4\pi} \int \vec{\nabla}_1 \cdot \left(\vec{C}(\vec{r}_2) \vec{\nabla}_2 \frac{1}{r_{12}} \right) d\tau_2$$

$$* \nabla_1 f(\vec{r}_1 - \vec{r}_2) = -\nabla_2 f(\vec{r}_1 - \vec{r}_2)$$

$$= \frac{1}{4\pi} \int \vec{C}(\vec{r}_2) \cdot (\vec{\nabla}_2 \vec{\nabla}_1 \frac{1}{r_{12}}) d\tau_2 = \frac{1}{4\pi} \int \vec{C}(\vec{r}_2) \cdot (\vec{\nabla}_2 \cdot \vec{\nabla}_2 \frac{1}{r_{12}}) d\tau_2$$

$$dA_{\text{NP}}|_m = \frac{1}{4\pi} \int \vec{C}(\vec{r}_2) \cdot \vec{\nabla}_2 \frac{1}{r_{12}} d\vec{r}_2$$

$$= \frac{1}{4\pi} \int \vec{\nabla}_2 \cdot \left[\vec{C}(\vec{r}_2) \frac{\partial}{\partial n_2} \frac{1}{r_{12}} \right] d\vec{r}_2 - \frac{1}{4\pi} \int \underbrace{(\vec{\nabla}_2 \cdot \vec{C}(\vec{r}_2))}_{=0} \frac{\partial}{\partial n_2} \frac{1}{r_{12}} d\vec{r}_2$$

$$\stackrel{\text{Ost.}}{=} \frac{1}{4\pi} \int_{S \rightarrow \infty} \vec{C}(\vec{r}_2) \cdot \frac{\partial}{\partial n_2} \frac{1}{r_{12}} d\vec{\sigma}_2 \quad \underline{\underline{C_{100} = 0}} \quad 0$$

2. Coordinate systems (Arf 3rd ed)

2. Vector analysis in curved Coordinates and tensors

(Arf - web 5th ed)

$$\nabla^2 \psi + k^2 \psi = 0$$

→ دستگاهی که در آن این معادله به
معادله دیراسل ساده تجزیه شود

IF $k^2 = 0 \leadsto \nabla^2 \psi = 0$ Laplace equation

$k^2 > 0 \leadsto$ Helmholtz's $\checkmark$

$k^2 < 0 \leadsto$ Diffusion $\checkmark$
انتشار

$k^2 \propto \text{Potential energy} \Rightarrow \text{Schrödinger} \checkmark$

2. Curvilinear Coordinates

$$\vec{r} = \begin{cases} x = C_1 & \hat{e}_1 \\ y = C_2 & \hat{e}_2 \\ z = C_3 & \hat{e}_3 \end{cases} \quad \text{مختصات دکارتی}$$

$$q_1, q_2, q_3 = \vec{r} \Rightarrow \text{سطوح مختصات} \quad \begin{cases} q_1 = C_1 \\ q_2 = C_2 \\ q_3 = C_3 \end{cases}$$

$$\begin{aligned} (r, \theta, \varphi) &\rightarrow \begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \\ (u, y, z) &\rightarrow \begin{cases} r = \sqrt{u^2 + y^2 + z^2} \\ \varphi = \arctan\left(\frac{y}{u}\right) \\ \theta = \frac{z}{\sqrt{u^2 + y^2 + z^2}} \end{cases} \end{aligned}$$

$$\begin{cases} u = u(q_1, q_2, q_3) \\ y = y(q_1, q_2, q_3) \\ z = z(q_1, q_2, q_3) \end{cases}$$

$$\begin{cases} z = z(q_1, q_2, q_3) \\ q_1 = q_1(u, y, z) \end{cases}$$

این سطوح لزوماً عمود بر هم نیستند اما
دستگاه مختصات برای مطالعه می‌کنیم
که این سطوح عمود باشند

$$q_i = C_i \quad \text{محور به رد} = \hat{e}_i$$

مختصات $\hat{e}_i$ در جهت افزایش q_i است

$$r_1 = C_1 \quad \hat{e}_1 \rightarrow r_1 \text{ هم جهت افزایش } r_1$$

$$r_i = r_i(r_1, r_2, r_3)$$

$$r_2 = C_2 \quad \hat{e}_2 \rightarrow r_2 \sim \sim$$

$$r_3 = C_3 \quad \hat{e}_3 \rightarrow r_3 \sim \sim$$

- Line element in curved linear Coordinates

$$r_i = r_i(r_1, r_2, r_3), i=1,2,3$$

$$dr_i = \sum_{j=1}^3 \frac{\partial r_i}{\partial r_j} dr_j$$

$$ds^2 = dr^2 + dy^2 + dz^2 = \sum_{i=1}^3 dr_i dr_i = \sum_{i=1}^3 \left[\left(\sum_{j=1}^3 \frac{\partial r_i}{\partial r_j} dr_j \right) \left(\sum_{k=1}^3 \frac{\partial r_i}{\partial r_k} dr_k \right) \right]$$

↓
Cobination

$$= \sum_{j,k} \left[\left(\sum_i \frac{\partial r_i}{\partial r_j} \frac{\partial r_i}{\partial r_k} \right) dr_j dr_k \right]$$

$$g_{jk} = \sum_{i=1}^3 \frac{\partial r_i}{\partial r_j} \frac{\partial r_i}{\partial r_k} = g_{kj}$$

↳ metric

$$\rightarrow ds^2 = \sum_{ij} g_{ij} dr_i dr_j$$

دستگاه مختصات کروی

$$\begin{cases} g_{ij} = 0 & i \neq j \\ g_{ii} = h^2 & i = 1, 2, 3 \end{cases}$$

↳ $h = \text{scale factors}$

فاصله های مقیاس

$$ds^2 = g_{11} (dr_1)^2 + g_{22} (dr_2)^2 + g_{33} (dr_3)^2 + 2(g_{12} dr_1 dr_2 + g_{23} dr_2 dr_3 + g_{13} dr_1 dr_3)$$

$$\xrightarrow{\text{مقیاس ها}} ds^2 = \sum_{i=1}^3 g_{ii} (dr_i)^2 = (h_1 dr_1)^2 + (h_2 dr_2)^2 + (h_3 dr_3)^2$$

$$\rightarrow h_i^2 = \sum_{j=1}^3 \left(\frac{\partial x_j}{\partial q_i} \right)^2 = \left(\frac{\partial x}{\partial q_i} \right)^2 + \left(\frac{\partial y}{\partial q_i} \right)^2 + \left(\frac{\partial z}{\partial q_i} \right)^2$$

$$\text{اگر } q_1, q_2, q_3 \text{ مختصات ثابت باشد} \rightarrow dS_i = h_i dq_i$$

مثال ۱: CLC

$$\begin{cases} d\vec{s} = dx^2 + dy^2 + dz^2 \\ d\vec{r} = \hat{i} dx + \hat{j} dy + \hat{k} dz \end{cases}$$

$$\text{CLC} \begin{cases} dS^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2 \\ d\vec{r} = \hat{e}_1 h_1 dq_1 + \hat{e}_2 h_2 dq_2 + \hat{e}_3 h_3 dq_3 \end{cases} \quad (1)$$

$$\vec{v} = \hat{e}_1 v_1 + \hat{e}_2 v_2 + \hat{e}_3 v_3 \quad (2)$$

$$\int \vec{v} \cdot d\vec{r} = \int v_1 h_1 dq_1 + \int v_2 h_2 dq_2 + \int v_3 h_3 dq_3 \quad (3)$$

$$d\vec{r} = \sum_{i=1}^3 \hat{e}_i h_i dq_i \xrightarrow{\div h_j dq_j} \frac{1}{h_j} \frac{\partial \vec{r}}{\partial q_j} = \sum_{i=1}^3 \frac{\hat{e}_i h_i}{h_j} \frac{dq_i}{dq_j} = \sum_{i=1}^3 \hat{e}_i \frac{h_i}{h_j} \cdot \delta_{ij} = \sum_{i=1}^3 \hat{e}_i$$

$$= \hat{e}_j \rightarrow \hat{e}_j = \frac{1}{h_j} \left[\hat{i} \frac{\partial x}{\partial q_j} + \hat{j} \frac{\partial y}{\partial q_j} + \hat{k} \frac{\partial z}{\partial q_j} \right]$$

مثال ۲: CLC $\rightarrow \hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

$$d\vec{\sigma}_1 = \hat{e}_1 dS_1 dS_3 = \hat{e}_1 \cdot h_2 h_3 dq_2 dq_3 = \hat{e}_3 \hat{e}_2 \text{ موازی } q_1 = C \text{ عمود بر}$$

$$d\vec{\sigma}_2 = \hat{e}_2 dS_1 dS_3 = \hat{e}_2 \cdot h_1 h_3 dq_1 dq_3 = \hat{e}_3 \hat{e}_1 \text{ موازی } q_2 = C \text{ عمود بر}$$

$$d\vec{\sigma}_3 = \hat{e}_3 dS_1 dS_2 = \hat{e}_3 \cdot h_1 h_2 dq_1 dq_2 = \hat{e}_1 \hat{e}_2 \text{ موازی } q_3 = C \text{ عمود بر}$$

$$d\vec{\sigma} = d\vec{\sigma}_1 + d\vec{\sigma}_2 + d\vec{\sigma}_3$$

$$= \hat{e}_1 h_2 h_3 dq_2 dq_3 + \hat{e}_2 h_1 h_3 dq_1 dq_3 + \hat{e}_3 h_1 h_2 dq_1 dq_2 \quad (4)$$

$$\textcircled{2}, \textcircled{\alpha}: \int_S \vec{V} \cdot d\vec{S} = \int_S V_1 h_2 h_3 dr_2 dr_3 + \int_S V_2 h_1 h_3 dr_1 dr_3 + \int_S V_3 h_1 h_2 dr_1 dr_2$$

$$* V_i = V_i(r_1, r_2, r_3)$$

انکڑال حصی در دستگاہ / مقام

$$d\mathcal{V} = dS_1 dS_2 dS_3 = h_1 h_2 h_3 dr_1 dr_2 dr_3$$

$$d\mathcal{V} = \sqrt{|g|} d\vec{r} \rightarrow |g| = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \stackrel{\text{مقام}}{=} \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$$

ماتریس متریک

$$= \sqrt{h_1^2 h_2^2 h_3^2} dr_1 dr_2 dr_3$$

$$= h_1 h_2 h_3 dr_1 dr_2 dr_3$$

$$\rightarrow \int_V \varphi(r_1, r_2, r_3) d\mathcal{V}$$

$$= \int_V \varphi(r_1, r_2, r_3) h_1 h_2 h_3 dr_1 dr_2 dr_3$$

ضرب داخلی در CLC مقام:

$$\left\{ \begin{array}{l} \vec{A} = \hat{e}_1 A_1 + \hat{e}_2 A_2 + \hat{e}_3 A_3 \\ \vec{B} = \hat{e}_1 B_1 + \hat{e}_2 B_2 + \hat{e}_3 B_3 \end{array} \right\} \vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$\left\{ \begin{array}{l} \hat{e}_1 \times \hat{e}_2 = \hat{e}_3 \\ \hat{e}_2 \times \hat{e}_3 = \hat{e}_1 \\ \hat{e}_3 \times \hat{e}_1 = \hat{e}_2 \end{array} \right.$$



$$\left(\vec{A} \times \vec{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \right)$$

2.2 Differential operators in CLC

- Gradient

$$ds_i = h_i dr_i$$

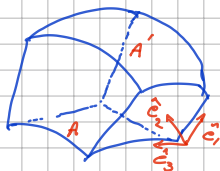
$$\vec{\nabla} \varphi|_i = \frac{\partial \varphi}{\partial s_i} = \frac{1}{h_i} \frac{\partial \varphi}{\partial r_i}$$

$$\rightarrow \vec{\nabla} \varphi = \sum_{i=1}^3 \frac{\hat{e}_i}{h_i} \frac{\partial \varphi}{\partial r_i} = \frac{\hat{e}_1}{h_1} \frac{\partial \varphi}{\partial r_1} + \frac{\hat{e}_2}{h_2} \frac{\partial \varphi}{\partial r_2} + \frac{\hat{e}_3}{h_3} \frac{\partial \varphi}{\partial r_3}$$

- Divergence.

$$\vec{V} = \vec{V}(r_1, r_2, r_3)$$

$$\vec{\nabla} \cdot \vec{V} = \lim_{\Delta \tau \rightarrow 0} \frac{\int \vec{V} \cdot d\vec{\sigma}}{\int d\tau}$$



$$\vec{\nabla} \cdot \vec{V} = \lim_{\Delta \tau \rightarrow 0} \frac{\int \vec{V} \cdot d\vec{\sigma}}{\int d\tau} = \vec{V} \cdot d\vec{\sigma}|_{A'} - \vec{V} \cdot d\vec{\sigma}|_A = \left[\cancel{\vec{V} \cdot d\vec{\sigma}|_A} + dr_1 \frac{\partial}{\partial r_1} (V \cdot d\sigma) - \cancel{V \cdot d\sigma}|_A \right]$$

$$= dr_1 \frac{\partial}{\partial r_1} (V_1 h_2 h_3) dr_2 dr_3$$

$$= \frac{\partial}{\partial r_1} (V_1 h_2 h_3) dr_1 dr_2 dr_3$$

$$\vec{\nabla} \cdot \vec{V} = \left[\frac{\partial}{\partial r_1} (V_1 h_2 h_3) + \frac{\partial}{\partial r_2} (V_2 h_1 h_3) + \frac{\partial}{\partial r_3} (V_3 h_2 h_1) \right] \frac{dr_1 dr_2 dr_3 \times h_1 h_2 h_3}{h_1 h_2 h_3}$$

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial r_1} (V_1 h_2 h_3) + \frac{\partial}{\partial r_2} (V_2 h_1 h_3) + \frac{\partial}{\partial r_3} (V_3 h_2 h_1) \right]$$

$$\vec{V} = \hat{e}_1 v_1 + \hat{e}_2 v_2 + \hat{e}_3 v_3 \quad (1) \checkmark$$

$$\vec{\nabla} \psi = \hat{e}_1 \frac{1}{h_1} \frac{\partial \psi}{\partial r_1} + \hat{e}_2 \frac{1}{h_2} \frac{\partial \psi}{\partial r_2} + \hat{e}_3 \frac{1}{h_3} \frac{\partial \psi}{\partial r_3} \quad (2) \checkmark$$

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial r_1} (v_1 h_2 h_3) + \frac{\partial}{\partial r_2} (v_2 h_1 h_3) + \frac{\partial}{\partial r_3} (v_3 h_1 h_2) \right] \quad (3) \checkmark$$

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial r_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial r_1} \right) + \frac{\partial}{\partial r_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \psi}{\partial r_2} \right) + \frac{\partial}{\partial r_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial r_3} \right) \right] \quad (4) \checkmark$$

$$\vec{\nabla} \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h_3 \\ \frac{\partial}{\partial r_1} & \frac{\partial}{\partial r_2} & \frac{\partial}{\partial r_3} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix} \quad (5) \rightarrow \text{رشت به دست آوردن ضرب}$$

$$h_i^2 = \left(\frac{\partial x}{\partial r_i} \right)^2 + \left(\frac{\partial y}{\partial r_i} \right)^2 + \left(\frac{\partial z}{\partial r_i} \right)^2 \quad (6)$$

$$d\vec{r} = \sum_i (\hat{e}_i h_i dr_i) \quad (7)$$

$$d\vec{V} = \hat{e}_1 h_2 h_3 dr_2 dr_3 + \hat{e}_2 h_1 h_3 dr_1 dr_3 + \hat{e}_3 h_1 h_2 dr_1 dr_2 \quad (8)$$

$$dV = h_1 h_2 h_3 dr_1 dr_2 dr_3 \quad (9)$$

$$\vec{\nabla} = \hat{e}_1 \frac{1}{h_1} \frac{\partial}{\partial r_1} + \hat{e}_2 \frac{1}{h_2} \frac{\partial}{\partial r_2} + \hat{e}_3 \frac{1}{h_3} \frac{\partial}{\partial r_3} \quad (10)$$

$$\hat{e}_i = \frac{1}{h_i} \left[\hat{i} \frac{\partial x}{\partial r_i} + \hat{j} \frac{\partial y}{\partial r_i} + \hat{k} \frac{\partial z}{\partial r_i} \right]$$

$$\vec{\nabla} \times \vec{w} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

2.3 Special coordinate Systems :

- Rectangular Cartesian System.

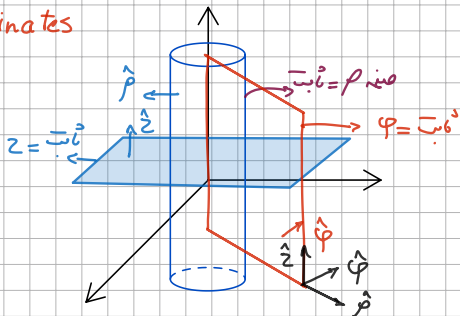
$$u_i = u_i(r_1, r_2, r_3)$$

$$\rightarrow \left\{ \begin{array}{l} x=r_1, y=r_2, z=r_3 \\ \hat{e}_1=\hat{i}, \hat{e}_2=\hat{j}, \hat{e}_3=\hat{k} \\ h_1=h_2=h_3=1 \end{array} \right\} \rightarrow \begin{array}{l} \text{بجائزہ در فیصلہ} \\ \text{زیر می رسم} \\ \text{دھان فیصلہ} \\ \text{اول می رسم} \end{array}$$

$\nabla^2 \psi, \vec{\nabla}_x \vec{V}, \vec{\nabla} \cdot \vec{V}, \vec{\nabla} \psi$

2.4 Circular Cylindrical Coordinates

$$\left. \begin{aligned} x &= \rho \cos \varphi \\ y &= \rho \sin \varphi \\ z &= z \end{aligned} \right\} \begin{cases} q_1 = \rho \\ q_2 = \varphi \\ q_3 = z \end{cases}$$



هر سه متغیر $z = C_3, \rho = C_2, \varphi = C_1$

$$h_1^2 = h_\rho^2 = \left(\frac{\partial x}{\partial \rho} \right)^2 + \left(\frac{\partial y}{\partial \rho} \right)^2 + \left(\frac{\partial z}{\partial \rho} \right)^2 = 1 \rightarrow h_1 = 1$$

$$h_2^2 = h_\varphi^2 = -\rho^2 \sin^2 \varphi + \rho^2 \cos^2 \varphi + 0 = \rho^2 \rightarrow h_2 = \rho$$

$$h_3^2 = h_z^2 = 0 + 0 + 1 = 1 \rightarrow h_3 = 1$$

$$d\vec{r} = \sum_i (\hat{e}_i h_i dq_i) = \hat{\rho} d\rho + \hat{\varphi} \rho d\varphi + \hat{k} dz$$

$$d\vec{r} = \hat{\rho} \rho d\varphi dz + \hat{\varphi} \rho d\rho dz + \hat{k} \rho d\rho d\varphi$$

$$\vec{\nabla} \psi = \hat{\rho} \frac{\partial \psi}{\partial \rho} + \hat{\varphi} \frac{1}{\rho} \frac{\partial \psi}{\partial \varphi} + \hat{k} \frac{\partial \psi}{\partial z}$$

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho V_\rho) + \frac{\partial}{\partial \varphi} (V_\varphi) + \frac{\partial}{\partial z} (\rho V_z) \right] = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho V_\rho) + \frac{1}{\rho} \frac{\partial V_\varphi}{\partial \varphi} + \frac{\partial V_z}{\partial z}$$

$$\nabla^2 \psi = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{\rho} \frac{\partial \psi}{\partial \varphi} \right) + \frac{\partial}{\partial z} \left(\rho \frac{\partial \psi}{\partial z} \right) \right]$$

$$= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{\rho} & \hat{\varphi} \rho & \hat{k} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\varphi & V_z \end{vmatrix} \quad \left\{ \begin{aligned} \hat{\rho} &= \hat{i} \cos \varphi + \hat{j} \sin \varphi \\ \hat{\varphi} &= \frac{1}{\rho} (-\hat{i} \rho \sin \varphi + \hat{j} \rho \cos \varphi) = -\hat{i} \sin \varphi + \hat{j} \cos \varphi \\ \hat{k} &= \hat{k} \end{aligned} \right.$$

$$\vec{\nabla}^2 \vec{V} = \nabla^2 (\hat{\rho} V_\rho + \hat{\varphi} V_\varphi + \hat{k} V_z)$$

$$\vec{\nabla}_x (\vec{\nabla}_x \vec{V}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{V}) - \nabla^2 \vec{V}$$

$$\nabla^2 \vec{V} = \vec{\nabla} (\vec{\nabla} \cdot \vec{V}) - \underbrace{\vec{\nabla}_x (\vec{\nabla}_x \vec{V})}_{\vec{w}}$$

$$\nabla^2 \vec{V}|_\rho = \nabla^2 V_\rho - \frac{1}{\rho^2} V_\rho - \frac{2}{\rho^2} \frac{\partial V_\varphi}{\partial \varphi}$$

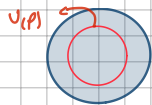
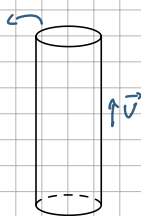
$$\nabla^2 \vec{V}|_\varphi = \nabla^2 V_\varphi - \frac{1}{\rho^2} V_\varphi + \frac{2}{\rho^2} \frac{\partial V_\rho}{\partial \varphi}$$

$$\nabla^2 \vec{V}|_z = \nabla^2 V_z$$

Example 2.4.1

$$\vec{\nabla}_x [\vec{\nabla}_x (\vec{\nabla}_x \vec{V})] = ?$$

$$\vec{V} = \hat{k} V(\rho)$$



$$V_\varphi = V_\rho = 0 \leadsto \vec{\nabla}_x \vec{V} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\varphi} & \hat{k} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ 0 & 0 & V_z(\rho) \end{vmatrix}$$

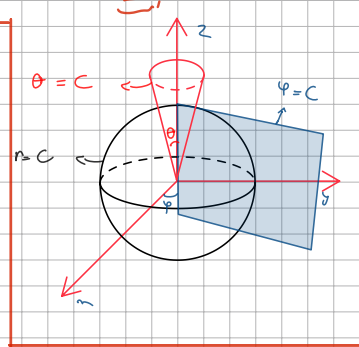
$$= -\hat{\varphi} \left(\frac{\partial V_z(\rho)}{\partial \rho} \right) = \vec{\omega}$$

$$\vec{\nabla}_x \vec{\omega} = \begin{vmatrix} \hat{\rho} & \rho \hat{\varphi} & \hat{k} \\ 0 & 0 & V_z \\ 0 & -\frac{\partial V_z}{\partial \rho} & 0 \end{vmatrix} = +\hat{\rho} V_z \frac{\partial V_z(\rho)}{\partial \rho} = \vec{U}$$

$$\vec{\nabla}_x \vec{U} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\varphi} & \hat{k} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ V_z \frac{\partial V_z}{\partial \rho} & 0 & 0 \end{vmatrix} = \vec{0}$$

2.5 Spherical and polar coordinates

$$\left. \begin{aligned} r_1 &= r \\ r_2 &= \theta \\ r_3 &= \varphi \end{aligned} \right\} \left. \begin{aligned} x &= r \sin \theta \cos \varphi \\ y &= r \sin \theta \sin \varphi \\ z &= r \cos \theta \end{aligned} \right\} \left. \begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \rightarrow \text{المسافة عن نقطة الأصل} \\ \varphi &= \text{Arctan} \left(\frac{y}{x} \right) \rightarrow \text{زاوية في المستوى } xy \\ \theta &= \text{Arccos} \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) \rightarrow \text{زاوية مع المحور } z \end{aligned} \right.$$



$$h_1 = h_r = \sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + \sin^2 \theta = 1 \rightarrow h_r = 1$$

$$h_2 = h_\theta = r^2 \cos^2 \theta \cos^2 \varphi + r^2 \cos^2 \theta \sin^2 \varphi + r^2 \sin^2 \theta = r^2 \rightarrow h_\theta = r$$

$$h_3 = h_\varphi = r^2 \sin^2 \theta \sin^2 \varphi + r^2 \sin^2 \theta \cos^2 \varphi = r^2 \sin^2 \theta \rightarrow h_\varphi = r \sin \theta$$

$$d\vec{r} = \hat{r} d\theta + \hat{\theta} r d\theta + \hat{\varphi} r \sin \theta d\varphi$$

$$\vec{r} = \hat{r} r \rightarrow d\vec{r} = \hat{r} dr + \hat{\theta} r d\theta + \hat{\varphi} r \sin \theta d\varphi$$

$$\rightarrow dr = \hat{\theta} d\theta + \hat{\varphi} \sin \theta d\varphi$$

$$\vec{e}_i = \frac{\partial \vec{r}}{\partial x_i} = \frac{\partial}{\partial x_i} (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta)$$

$$d\vec{r} = \sum_i \vec{e}_i \frac{h_i}{h_i} \frac{\partial r}{\partial x_i} dx_i = \frac{h_i}{h_i} \frac{\partial r}{\partial x_i} dx_i = \hat{r} r^2 \sin \theta d\theta d\varphi + \hat{\theta} r \sin \theta dr d\varphi + \hat{\varphi} r dr d\theta$$

$$d\Omega = \sin \theta d\theta d\varphi \Rightarrow \int_{\text{all}} d\Omega = \int_0^{2\pi} \int_0^\pi \sin \theta d\theta d\varphi = 4\pi$$

$$\text{if } dr=0 \rightarrow d\vec{r} = \hat{r} r^2 \sin \theta d\theta d\varphi = \hat{r} r^2 d\Omega$$

$$\text{if } d\theta=0 \rightarrow d\vec{r} = \hat{\theta} r \sin \theta dr d\varphi$$

$$\text{if } d\varphi=0 \rightarrow d\vec{r} = \hat{\varphi} r dr d\theta$$

$$d\Omega = h_r h_\theta h_\varphi dr d\theta d\varphi = r^2 \sin \theta dr d\theta d\varphi$$

$$\vec{e}_i = \frac{1}{h_i} \sum_j \vec{e}_j \frac{\partial h_j}{\partial x_i}$$

$$\hat{e}_r = \hat{r} = \hat{i} \frac{x}{r} + \hat{j} \frac{y}{r} + \hat{k} \frac{z}{r} = \hat{i} \sin \theta \cos \varphi + \hat{j} \sin \theta \sin \varphi + \hat{k} \cos \theta$$

$$\hat{e}_\theta = \hat{\theta} = \frac{1}{r} (\hat{i} r \cos \theta \cos \varphi + \hat{j} r \cos \theta \sin \varphi - \hat{k} r \sin \theta) = \hat{i} \cos \theta \cos \varphi + \hat{j} \cos \theta \sin \varphi - \hat{k} \sin \theta$$

$$\hat{e}_\varphi = \hat{\varphi} = \frac{1}{r \sin \theta} (-\hat{i} r \sin \theta \sin \varphi + \hat{j} r \sin \theta \cos \varphi) = -\hat{i} \sin \varphi + \hat{j} \cos \varphi + \hat{k} 0$$

$$\vec{v} = \sum_i \hat{e}_i v_i$$

$$\vec{\nabla} \psi = \sum_i \frac{\hat{e}_i}{h_i} \frac{\partial \psi}{\partial r_i}$$

$$\vec{v} \cdot \vec{v} = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial r_i} \left(\frac{v_i h_1 h_2 h_3}{h_i} \right)$$

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \sum_i \frac{\partial}{\partial r_i} \left(\frac{h_1 h_2 h_3}{h_i^2} \frac{\partial \psi}{\partial r_i} \right)$$

$$\vec{\nabla}_x \vec{v} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h_3 \\ \frac{\partial}{\partial r_1} & \frac{\partial}{\partial r_2} & \frac{\partial}{\partial r_3} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix}$$

$$d\vec{r} = \sum_i \hat{e}_i d\alpha_i = \sum_i \hat{e}_i \frac{h_1 h_2 h_3}{h_i} \cdot \frac{dr_1 dr_2 dr_3}{dr_i}$$

$$dV = h_1 h_2 h_3 dr_1 dr_2 dr_3$$

$$\hat{e}_i = \frac{1}{h_i} \frac{\partial \vec{r}}{\partial r_i} = \frac{1}{h_i} \sum_j \hat{e}_j \frac{\partial h_j}{\partial r_i}$$

$$d\vec{r} = \sum_i \hat{e}_i h_i dr_i$$

$$g_{ij} = g_{ji} = \frac{h_i h_j}{h_i h_j} \quad \text{if } i \neq j$$

* فقط برای دو سه مختصات

$$\vec{\nabla}^2, \vec{\nabla}_x, \vec{\nabla}_\theta, \vec{\nabla}_\varphi$$

$$\vec{V} = \hat{r} V_r + \hat{\theta} V_\theta + \hat{\varphi} V_\varphi$$

$$\vec{\nabla} \psi = \hat{r} \frac{\partial \psi}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial \psi}{\partial \theta} + \frac{\hat{\varphi}}{r \sin \theta} \frac{\partial \psi}{\partial \varphi}$$

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r^2 \sin \theta} \left[\sin \theta \frac{\partial}{\partial r} (r^2 V_r) + r \frac{\partial}{\partial \theta} (\sin \theta V_\theta) + r \frac{\partial}{\partial \varphi} (V_\varphi) \right]$$

$$\vec{\nabla}^2 = \frac{1}{r^2 \sin \theta} \left[\sin \theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + r \frac{\partial}{\partial \theta} \left(\sin \theta \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right) + r \frac{\partial}{\partial \varphi} \left(\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \varphi} \right) \right]$$

$$= \frac{1}{r^2 \sin \theta} \left[\sin \theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 \psi}{\partial \varphi^2} \right]$$

$$\vec{\nabla}_x \vec{V} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r \hat{\theta} & r \sin \theta \hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ V_r & V_\theta & V_\varphi \end{vmatrix}$$

$$\vec{\nabla}^2 \vec{V} = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \vec{\nabla}_x(\vec{\nabla}_x \vec{V})$$

$$\psi = f(r) \rightarrow \vec{\nabla} \psi = \hat{r} \frac{df(r)}{dr}$$

$$\vec{V} = \hat{r} f(r) \rightarrow \vec{\nabla} \cdot \vec{V} = \frac{1}{r^2 \sin \theta} \sin \theta \frac{\partial}{\partial r} (r^2 f(r)) = \frac{1}{r^2} \frac{d}{dr} (r^2 f(r)) = \frac{2}{r} f(r) + \frac{df(r)}{dr}$$

$$\psi = f(r) \rightarrow \vec{\nabla}^2 \psi = \vec{\nabla}^2 f(r) = \frac{1}{r^2 \sin \theta} \sin \theta \frac{d}{dr} \left(r^2 \frac{df}{dr} \right) = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{df}{dr} \right) = \frac{2}{r} \frac{df}{dr} + \frac{d^2 f}{dr^2}$$

$$\vec{V} = \hat{r} f(r) \rightarrow \vec{\nabla}_x \vec{V} = \begin{vmatrix} \hat{r} & r \hat{\theta} & r \sin \theta \hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ f(r) & 0 & 0 \end{vmatrix} = 0$$

معین

$$\begin{cases} x = r \sin \theta \cosh \varphi \\ y = r \sin \theta \sinh \varphi \\ z = r \cosh \theta \end{cases} \quad \begin{cases} u = r \sinh \theta \cos \varphi \\ v = r \sinh \theta \sin \varphi \\ z = r \cosh \theta \end{cases} \quad \begin{cases} u = r \sinh \theta \cosh \varphi \\ v = r \sinh \theta \sinh \varphi \\ z = r \cosh \theta \end{cases}$$

تمام کردی نه برای کوه کردن را برای این سه نیز انجام ده

Exp 25.2:

$$\vec{V} = \vec{\nabla} \times [\vec{\nabla} \times (\hat{\varphi} A_{\varphi}(r, \theta))]$$

$$\vec{A} = \hat{\varphi} A_{\varphi}(r, \theta)$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ 0 & 0 & r\sin\theta A_{\varphi} \end{vmatrix} = \frac{1}{r^2 \sin \theta} \left(\hat{r} \frac{\partial(r\sin\theta A_{\varphi})}{\partial \theta} - \hat{\theta} r \frac{\partial}{\partial r}(r\sin\theta A_{\varphi}) \right) = \hat{r} B_r + \hat{\theta} B_{\theta}$$

$$\vec{V} = \vec{\nabla} \times \vec{B} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ B_r & B_{\theta} & 0 \end{vmatrix} = -\hat{\varphi} \left[\nabla^2 A_{\varphi} - \frac{1}{r^2 \sin^2 \theta} A_{\varphi} \right]$$

$$\vec{E} = -\vec{\nabla} \varphi - \frac{\partial \vec{A}}{\partial t}$$

26 Separation of variables

$$\nabla^2 \psi + k^2 \psi = 0$$

Cartezian :

$$\psi(x, y, z) = X(x) Y(y) Z(z)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0$$

$$Y Z \frac{d^2 X}{dx^2} + X Z \frac{d^2 Y}{dy^2} + X Y \frac{d^2 Z}{dz^2} + k^2 X Y Z = 0$$

$$\rightarrow \frac{X Z}{X} \frac{1}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} + k^2 = 0 \rightarrow \frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} - \frac{1}{Z} \frac{d^2 Z}{dz^2} - k^2 = -l^2$$

$$\rightarrow \frac{1}{X} \frac{d^2 X}{dx^2} = -l^2$$

$$\rightarrow \frac{1}{Y} \frac{d^2 Y}{dy^2} = l^2 - k^2 - \frac{1}{Z} \frac{d^2 Z}{dz^2} = -m^2 \rightarrow \frac{1}{Y} \frac{d^2 Y}{dy^2} = -m^2$$

$$\rightarrow \frac{1}{Z} \frac{d^2 Z}{dz^2} = l^2 + m^2 - k^2 = -n^2$$

$$\psi(x, y, z) = X_l(x) Y_m(y) Z_n(z) = \psi_{lmn}(x, y, z)$$

$$\psi(x, y, z) = \sum_{l, m, n} a_{lmn} X_l(x) Y_m(y) Z_n(z)$$

Linear operation

$$L(a_1 \psi_1 + a_2 \psi_2) = a_1 L(\psi_1) + a_2 L(\psi_2)$$

$\Rightarrow L = \text{Linear operation}$

$$(\nabla^2 + k^2) \left(\sum_{l, m, n} a_{lmn} X_l Y_m Z_n \right)$$

$$\rightarrow \sum_{l, m, n} (\nabla^2 + k^2) (a_{lmn} X_l Y_m Z_n) = \sum_{l, m, n} a_{lmn} \underbrace{(\nabla^2 + k^2) X_l Y_m Z_n}_0 = 0$$

نتیجه

معادله برای هر یک از متغیرها به صورت زیر است :

$$[\nabla^2 + f(x) + g(y) + h(z) + k^2] \psi(x, y, z) = 0$$

جداری مرتبه ای

$$(\nabla^2 + k^2) \psi(\rho, \varphi, z) = 0$$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0$$

$$\psi = P(\rho) \Phi(\varphi) Z(z)$$

$$\rightarrow \frac{\Phi Z}{\rho} \frac{d}{d\rho} \left(\rho \frac{dP}{d\rho} \right) + \frac{P Z}{\rho^2} \frac{d^2 \Phi}{d\varphi^2} + P \Phi \frac{d^2 Z}{dz^2} + P \Phi Z k^2 = 0$$

$$\frac{-P \Phi Z}{\rho} \frac{1}{P} \frac{d}{d\rho} \left(\rho \frac{dP}{d\rho} \right) + \frac{1}{\rho^2 \Phi} \frac{d^2 \Phi}{d\varphi^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} + k^2 = 0$$

$$\rho \left(\frac{d^2 Z}{dz^2} + (k^2 - k_z^2) Z \right) = 0 \rightarrow Z = Z_n(z)$$

$$\frac{d^2 \Phi}{d\varphi^2} + m^2 \Phi = 0 \rightarrow \Phi = \Phi_m(\varphi)$$

$$\rho \frac{d}{d\rho} \left(\rho \frac{dP}{d\rho} \right) + (k^2 \rho^2 - m^2) P = 0 \rightarrow P = P_m(\rho)$$

$$\psi = \psi_{mn}(\rho, \varphi, z) = P_m(\rho) \Phi_m(\varphi) Z_n(z)$$

$$\psi(\rho, \varphi, z) = \sum_{mn} a_{mn} P_m \Phi_m Z_n$$

جداری مرتبه ای

روش انتگرالی فصلی و تبدیل از صفحه 163 به دیوان کتاب آملین جدول این است و بخوانید

$$\frac{d^2 \Phi}{d\varphi^2} + \Phi m^2 = 0 \rightarrow \Phi = \Phi_m(\varphi)$$

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + (k^2 r^2 - Q) R = 0 \rightarrow R = R_Q(r)$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) - \left(\frac{m^2}{\sin^2 \theta} - Q \right) \Theta = 0 \rightarrow \Theta = \Theta_{mQ}(\theta)$$

$$\left\{ \begin{array}{l} \psi_{mQ} = R_Q \oplus \Theta_{mQ} \Phi_m \\ \psi = \sum_{mQ} a_{mQ} R_Q \oplus \Theta_{mQ} \Phi_m \end{array} \right.$$

نشان دهید که روش فوق معادله را برقرار می‌کند

$$\nabla^2 + k^2 \left[f(r) + \frac{1}{r^2} g(r) + \frac{1}{r^2 \sin^2 \theta} h(\varphi) + k^2 \right] \psi(r, \theta, \varphi) = 0$$

3 Tensor analysis

3.1 Definitions

$$\vec{J} = \sigma' \vec{E} = \text{دائیه جریان}$$

$$\sigma' = \text{رسانش}$$

$$\vec{E} = \text{میدان الکتریکی}$$

$$\vec{J} = \sigma' E_i$$

isotropic یکنواخت

$$\vec{J}_i = \sum_j \sigma'_{ij} E_j \rightsquigarrow \sigma'_{ij} = \text{تانسور مرتبه دوم}$$

الترجیح همسانگرد نیافته

$$\vec{A} = B \vec{C}$$

الترجیح $\vec{A}$ و $\vec{C}$ موازی نباشند

$$\vec{A}_i = \sum_j B_{ij} C_j$$

$$\phi' = \phi \rightarrow \text{تانسور مرتبه صفر}$$

$$A'_i = \sum_j a_{ij} A_j$$

(تانسور مرتبه 1)

تانسور یک شیء ریاضی که تحت تبدیلات مختصات (موضوع، انتقال و...) مانند های آن طبق قانون خاصی تبدیل نشود

ن

تکرار دارد ابرین / مختصات بالایی داشته

$$\left. \begin{aligned} A' &\xrightarrow[\text{مختصات}]{\text{تغییر تبدیل}} A'' = \sum_i \frac{\partial x''^i}{\partial x'^i} A^i \\ B_i &\xrightarrow{''} B'_i = \sum_j \frac{\partial x^j}{\partial x''^i} B_j \end{aligned} \right\} \frac{\partial x''^i}{\partial x'^i} = \frac{\partial x^j}{\partial x''^i}$$

$$\text{rank} = 2 \leftarrow A^i \xrightarrow{''} A''^i = \sum_{m,n} \frac{\partial x''^i}{\partial x^m} \frac{\partial x^m}{\partial x''^n} A^n$$

$$\text{Type} = (2,0)$$

$$\text{rank} = 2$$

$$\text{Type} = (1,1)$$

$$B'_j \xrightarrow{''} B''_j = \sum_{m,n} \frac{\partial x''^j}{\partial x^m} \frac{\partial x^m}{\partial x''^n} B^n$$

rank = 2

Type = (0, 2)

$$C_{ij} \longrightarrow C'_{ij} = \sum_{mn} \frac{\partial x^m}{\partial x'^i} \frac{\partial x^n}{\partial x'^j} C^{mn}$$

$$A^{i_1, i_2, \dots, i_m}_{j_1, j_2, \dots, j_n} \xrightarrow{\sim} A'^{i_1, i_2, \dots, i_m}_{j_1, j_2, \dots, j_n} = \sum_{\substack{l_1, l_2, \dots, l_m \\ k_1, k_2, \dots, k_n}} \frac{\partial x^{l_1}}{\partial x'^{k_1}} \frac{\partial x^{l_2}}{\partial x'^{k_2}} \dots \frac{\partial x^{l_m}}{\partial x'^{k_m}} \frac{\partial x^{k_1}}{\partial x'^{j_1}} \frac{\partial x^{k_2}}{\partial x'^{j_2}} \dots \frac{\partial x^{k_n}}{\partial x'^{j_n}} A^{i_1, i_2, \dots, i_m}_{l_1, l_2, \dots, l_m, k_1, k_2, \dots, k_n}$$

↓
Rank = m + n

Type = (m, n)